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# On repeated application of the Epsilon algorithm.

( Chiffres, 4 (1961), p 19-22).

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# On Repeated Application of the ε-Algorithm

by P. Wynn.

Dans certains cas, la répétition de l'ε-Algorithme accélère efficacement la convergence d'une série lentement convergente. Cette technique est illustrée par un exemple.

In bestimmten Fällen beschleunigt die Wiederholung des ε-Algorithmus wirksam die Konvergenz einer langsam konvergierenden Reihe. Dieses Verfahren wird durch ein Beispiel veranschaulicht.

In certain cases, the repetition of the ε-Algorithm greatly accelerates the convergence of a slowly convergent sequence. This technique is illustrated by an example.

В некоторых случаях, повторение ε-алгорифм существенно ускоряет сходимость медленно сходящихся рядов. Дастся пример этого способа.

The purpose of this note is to draw attention to, and place upon record an example of, a powerful technique for the numerical transformation of slowly convergent sequences. It is mathematically equivalent to a proposal of Shanks [6].

As has been shown (see for example [1] and [2]) the epsilon algorithm [3] :

$$\varepsilon_{s+1}^{(m)} = \varepsilon_{s+1}^{(m+1)} + \frac{1}{\varepsilon_s^{(m+1)} - \varepsilon_s^{(m)}} \qquad (1)$$

with the initial conditions :

$$\varepsilon_{-1}^{(m)} = 0 \qquad \varepsilon_0^{(m)} = S_m \qquad m = 0, 1, \ldots \qquad (2)$$

is, in certain cases, a powerful method for the numerical transformation of a slowly convergent (or divergent) sequence $S_m$ $m = 0, 1, \ldots$, in the sense that the quantities $\varepsilon_{2s}^{(0)}$ $s = 0, 1, \ldots$ converge more rapidly to a limit with which the sequence $S_m$ may be associated than do the quantities $S_m$ $m = 0, 1, \ldots$

The quantities $\varepsilon_{2s}^{(m)}$ produced from the sequence $S_m$ may be referred to as ${}_1\varepsilon_{2s}^{(m)}$; it is then formally possible to apply the relationships (1) to derive sequence of quantities ${}_2\varepsilon_s^{(m)}$ $m, s = 0, 1, \ldots$ from the initial conditions :

$${}_2\varepsilon_{-1}^{(m)} = 0 \qquad {}_2\varepsilon_0^{(m)} = {}_1\varepsilon_{2m}^{(0)} \qquad m = 0, 1, \ldots; \tag{3}$$

and further to repeat this process. There is a limit to the number of times the process may be repeated. From $2n - 1$ $n = 1, 2, \ldots$ members of the sequence $\varepsilon_0^{(m)}$ $m = 0, 1, \ldots$ there are produced $n$ members of the sequence $\varepsilon_{2m}^{(0)}$ $m = 0, 1, \ldots$; the number of starting values is therefore approximately halved at each stage.

The numerical example concerns the transformation of the sequence of partial sums of the series :

$$- e^z \, \mathrm{E}\,i(-z) = e^z \int_z^\infty e^{-t} \, t^{-1} \, dt$$

$$\sim \sum_{n=0}^{\infty} (-1)^n \, n! \, z^{-n-1} \tag{4}$$

when $z = 0.1$. The series for this value of $z$ is :

$$0 + 10 - 100 + 2{,}000 - 60{,}000 + 2{,}400{,}000 - 120{,}000{,}000 \ldots; \tag{5}$$

the partial sums of the series are consecutively

$$0;\ 10;\ -90;\ +1{,}910;\ -58{,}090;\ +2{,}341{,}910;\ -117{,}658{,}090;\ \ldots \tag{6}$$

Table I gives the quantities ${}_k\varepsilon_{2s}^{(0)}$ $k = 1(1)4$, $s = 0, 1, \ldots, 2^{4-k} + 1$

| $s/k$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | 0.0000 000 | 0.0000 000 | 0.0000 000 | 0.0000 000 |
| 1 | 0.9090 909 | 1.5538 362 | 1.9841 277 | 2.0146 999 |
| 2 | 1.2863 070 | 1.8908 117 | 2.0142 360 | |
| 3 | 1.4966 417 | 1.9766 953 | | |
| 4 | 1.6295 095 | 2.0018 881 | | |
| 5 | 1.7196 014 | | | |
| 6 | 1.7835 970 | | | |
| 7 | 1.8305 965 | | | |
| 8 | 1.8659 999 | | | |

TABLE I

The value of $-e^{z}\mathrm{Ei}\,(-z)$ may be computed for small value of $z$ by use of the expression

$$-\,e^{z}\ \mathrm{E}i(-z) = -\,e^{z}\left(\gamma + \log_{e} z + \sum_{n=0}^{\infty} \frac{(-z)^{n}}{n\,!\,n}\right) \tag{7}$$

which gives, when $z = 0.1$,

$$-\,e^{0.1}\mathrm{Ei}(-\,0.1) = 2.0146\ 425 \tag{8}$$

It might be thought, in view of the size of the members of the sequence (6) and the relatively small magnitude of the quantities ${}_{1}\varepsilon^{(0)}_{2s}$, that the process of obtaining the quantities ${}_{1}\varepsilon^{(m)}_{s}$ would involve a catastrophic loss of figures due to cancellation. This is not so. Application of the linear error analysis given in [4] shows that the errors tend consistently to cancel each other.

It may be shown [5] that the ε-algorithm transforms the successive partial sums of the series into the successive convergents of the continued fraction

$$\frac{1}{z+1-}\ \frac{1^{2}}{z+3-}\ \cdots\ \frac{\nu^{2}}{z+2\nu+1-}\ \cdots \tag{9}$$

and as such the members of the first column of Table I have, without loss accuracy, been computed.

The even order columns of the ε-array relating to the sequence (6), which have been computed using floating point fixed length (30 binary digit) arithmetic, are displayed in Table II.

| m\s | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| 0 | 0.0 | | | | | |
| 1 | $+1.0 \times 10^{1}$ | $+9.0909\ 091 \times 10^{-1}$ | | | | |
| 2 | $-9.0 \times 10^{1}$ | $+5.2380\ 953 \times 10^{0}$ | $+1.2863\ 071 \times 10^{0}$ | | | |
| 3 | $+1.91 \times 10^{3}$ | $-2.5483\ 871 \times 10^{1}$ | $+3.7972\ 771 \times 10^{0}$ | $+1.4966\ 421 \times 10^{0}$ | | |
| 4 | $-5.809 \times 10^{4}$ | $+4.4658\ 545 \times 10^{2}$ | $-1.0374\ 696 \times 10^{1}$ | $+3.1423\ 839 \times 10^{0}$ | $+1.6295\ 082$ | |
| 5 | $+2.34191 \times 10^{6}$ | $-1.1031\ 172 \times 10^{4}$ | $+1.6797\ 235 \times 10^{2}$ | $-4.7103\ 262 \times 10^{0}$ | $+2.7846\ 563$ | $+1.7195\ 816$ |
| 6 | $-1.1765\ 809 \times 10^{8}$ | $+3.7469\ 650 \times 10^{5}$ | $-3.4921\ 367 \times 10^{3}$ | $+7.9542\ 832 \times 10^{1}$ | $-2.0548\ 172$ | |
| 7 | $+7.0823\ 420 \times 10^{9}$ | $-1.6249\ 664 \times 10^{7}$ | $+1.0279\ 353 \times 10^{5}$ | $-1.4214\ 087 \times 10^{3}$ | | |
| 8 | $-4.9691\ 766 \times 10^{11}$ | $+8.6012\ 006 \times 10^{8}$ | $-3.9278\ 420 \times 10^{6}$ | | | |
| 9 | $+3.9823\ 083 \times 10^{13}$ | $-5.3840\ 708 \times 10^{10}$ | | | | |
| 10 | $-3.5889\ 770 \times 10^{15}$ | | | | | |

TABLE II

Comparison of the leading diagonal of Table II and the first column of Table I shows that a certain loss of figures does occur in effecting the ε-algorithm, but catastrophic is it not.

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